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OPTIMAL DESIGN OF AXISYMMETRIC PLASTIC SHALLOW SHELLS OF VON MISES MATERIAL

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Abstract—An optimal design technique developed earlier for axisymmetric plates and circular cylindrical shells is accommodated for shallow spherical shells subjected to uniform transverse pressure. Material of the shells is assumed to be rigid–plastic obeying the von Mises yield condition and the associated deformation law. The post-yield behaviour of the shells is taken into account. The weight minimization is performed under the condition that the maximal deflection of the shell of variable thickness coincides with the deflection of the reference shell of constant thickness. The problem is transformed into a non-linear boundary value problem which is solved numerically.

NOTATION

A	radius of the shell
a, R	internal and external radii
M_0, N_0	limit moment and limit load
H	total thickness
h	thickness of carrying layers
V	material volume
U, W	displacements
σ_0	yield stress
ε_1, ε_2, κ_1, κ_2	strain components
N_1, N_2	membrane forces
M_1, M_2	bending moments
λ	scalar multiplier
$\Phi \leqslant 0$	yield condition
ρ, α, β	non-dimensional radii
$n_{1,2}$; $m_{1,2}$	non-dimensional stresses
u, w	non-dimensional displacements
v	non-dimensional thickness
p	non-dimensional load intensity
s	geometrical parameter
z	auxiliary variable
ψ_1–ψ_5, φ, μ, η	Lagrangian multipliers
e	economy coefficient

1. INTRODUCTION

The optimal design of thin walled shell structures has had the attention of many investigators during the last decades. The comprehensive surveys of this topic are presented by Kruzelecki and Zyczkowski (1985), also by Lellep and Lepik (1984). Since the minimum weight designs employ the material resources in the most efficient manner one may not neglect the plastic deformations which might occur during the exploitation of the optimized shell structures. This involves the need for the optimal design of plastic plates and shells. The early works on optimal design of plastic structures have used the geometrically linear concept, i.e. the designs have been determined for the given load carrying capacity. However, the work by Mróz and Gawecki (1975) revealed an unfavourable effect of the designs of this type—they might be sensitive to geometrical changes the structures undergo in the post-yield range. This means that the post-yield stiffness of the plate of variable thickness is smaller than that of the plate of constant thickness. Thus the geometrical changes should be taken into account in the statement of the problem when studying plastic shells of minimum weight.

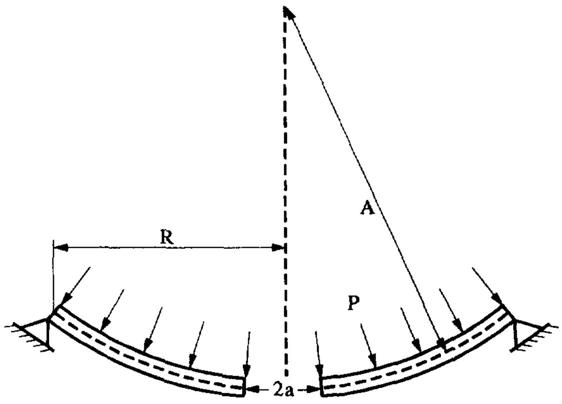

Fig. 1. Shell geometry.

Different approaches to the optimization of geometrically non-linear structures have been discussed by Lellep and Lepik (1984), Mróz and Gawecki (1975) and Lellep (1991). Making use of the variational methods of the optimal control theory in Lellep (1985) and Lellep and Sawczuk (1987), an optimization technique is developed for plastic cylindrical shells which operate in the post-yield range. Material of the structures is assumed to obey the Tresca yield condition. Shallow spherical shells of Tresca material have been studied by Lellep and Hein (1993) assuming the thickness of the cap is piecewise constant. Optimal design of geometrically non-linear rigid–plastic annular plates and cylindrical shells of von Mises material has been investigated by Lellep and Majak (1992, 1993) transforming the problem into a boundary value problem.

In the present paper a numerical procedure will be developed for minimization of the weight of plastic shallow spherical shells manufactured from a von Mises material. Geometrical non-linearity, i.e. moderately large deflections will be taken into account.

2. FORMULATION OF THE PROBLEM AND THE GOVERNING EQUATIONS

Let us consider a spherical cap of radius A subjected to the uniformly distributed internal pressure of intensity P (Fig. 1). The internal edge of the shell is free whilst the external edge of radius R may be pinned or simply supported. The radius of the central hole is denoted by a.

It is assumed that the shell wall is of ideal sandwich type, H being the constant total thickness. However, the thickness of carrying layers h is variable. Thus the yield moment $M_0 = \sigma_0 hH$ and the yield force $N_0 = 2\sigma_0 h$ are also functions of the radius r. Here σ_0 stands for the yield stress of the material of the carrying layers. The core material is assumed to be mildly resisting to the tangential shear stresses only.

Let us assume that the radius of the outer edge of the shell is small in comparison to the radius of the shell, i.e. $R \ll A$. In this case the material volume of the carrying layers

$$V = \int_S h \, dS, \tag{1}$$

where S is the central surface area, may be expressed as

$$V = 2\pi \int_a^R hr \, dr. \tag{2}$$

The approximation (2) of the exact material volume [eqn (1)] of shallow spherical shells is widely used in the literature. It will be employed in the present study also. We are looking for the minimum of eqn (2) under the requirement that the maximal deflection of the shell $W(a)$ coincides with the maximal deflection of the reference shell of constant thickness h_*.

It is assumed that the thickness of the carrying layers is constrained above. Thus, at each point the restriction

$$h - h_0 \leqslant 0 \tag{3}$$

is to be met. Here h_0 is a given constant. It is worthwhile to mention that the restriction (3) is imposed on the stress–strain state of the shell in order to get a unique solution of the optimization problem in the class of continuous functions. The optimal solution does not exist when eqn (3) is omitted in the statement of the problem. A similar situation occurred in the case of axisymmetric plates as previously shown by the authors (Lellep and Majak, 1993).

Let us assume that the strains are small but the displacements are finite. Thus the basic equations of the von Karman non-linear shell theory are applicable. The strain components corresponding to the von Karman theory may be presented as [see Sawczuk (1982)]

$$\varepsilon_1 = \frac{dU}{dr} + \frac{r}{A}\frac{dW}{dr} + \frac{1}{2}\left(\frac{dW}{dr}\right)^2, \quad \varepsilon_2 = \frac{U}{r},$$

$$\kappa_1 = -\frac{d^2W}{dr^2}, \quad \kappa_2 = -\frac{1}{r}\frac{dW}{dr}, \tag{4}$$

ε_1, ε_2 being the membrane strains and κ_1, κ_2 the curvatures. Here W and U stand for the transverse deflection and meridional displacement, respectively.

The equilibrium equations which take small configuration changes into account may be written as

$$\frac{d}{dr}(rN_1) - N_2 = 0, \quad \frac{d}{dr}\left\{\frac{d}{dr}(rM_1) - M_2 + rN_1\left(\frac{r}{A} + \frac{dW}{dr}\right)\right\} + Pr = 0, \tag{5}$$

where N_1, N_2 stand for the membrane forces and M_1, M_2 are the principal bending moments.

The material of the shells under consideration obeys the von Mises yield condition and associated deformation law. Parametrical equations of the yield surface in the space of the generalized stresses are derived by Ilioushine (1957). However, the exact yield surface is quite complicated. This involves the need for reasonable approximations of the exact yield surface corresponding to the von Mises yield condition.

In this paper the approximation of the yield surface which corresponds to the satisfaction of the original yield condition on the average will be utilized. The latter surface may be expressed as

$$\Phi = \frac{1}{N_0^2}(N_1^2 - N_1N_2 + N_2^2) + \frac{1}{M_0^2}(M_1^2 - M_1M_2 + M_2^2) \leqslant 0, \tag{6}$$

where M_0 and N_0 stand for the yield moment and yield force, respectively. It was shown by several authors [see, for example, Robinson (1971); Haydl and Sherbourne (1979)] that the non-linear approximation (6) of the exact yield surface gives reasonable results in the limit analysis of plastic plates and shells.

The associated deformation (gradientality) law states that

$$\varepsilon_i = \lambda\frac{\partial\Phi}{\partial N_i}, \quad \kappa_i = \lambda\frac{\partial\Phi}{\partial M_i}; \quad i = 1, 2 \tag{7}$$

if the deformation theory of plasticity is employed. It was shown by Budianski (1959) and Ponter and Martin (1972) that the deformation theory would provide a consistent approximation to the incremental solution. Here λ is a non-negative scalar multiplier and Φ is defined by eqn (6).

It is worth mentioning that $\lambda = 0$ in eqn (7) if $\Phi < 0$. This reflects the matter that the strain components [eqn (4)] must vanish if the stress-state corresponds to an interior point of the closed convex surface [eqn (6)]. If, however, $\Phi = 0$ the multiplier λ is a non-negative function.

It is easy to recheck that the relations (7) could be put into the form

$$\varepsilon_1 = \frac{\lambda}{N_0^2}(2N_1 - N_2), \quad \varepsilon_2 = \frac{\lambda}{N_0^2}(2N_2 - N_1),$$
$$\kappa_1 = \frac{\lambda}{M_0^2}(2M_1 - M_2), \quad \kappa_2 = \frac{\lambda}{M_0^2}(2M_2 - M_1). \tag{8}$$

Eliminating the strain components from the system [eqns (4) and (8)] one obtains the equations

$$\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{r}{A}\frac{\mathrm{d}W}{\mathrm{d}r} - \frac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 + \frac{1}{N_0^2}(2N_1 - N_2), \quad \frac{U}{r} = \frac{\lambda}{N_0^2}(2N_2 - N_1)$$
$$\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} = -\frac{\lambda}{M_0^2}(2M_1 - M_2), \quad \frac{\mathrm{d}W}{\mathrm{d}r} = -\frac{r\lambda}{M_0^2}(2M_2 - M_1). \tag{9}$$

It appears to be convenient to utilize the following non-dimensional quantities:

$$\rho = \frac{r}{A}, \quad \alpha = \frac{a}{A}, \quad \beta = \frac{R}{A}, \quad n_{1,2} = \frac{N_{1,2}}{N_*}, \quad m_{1,2} = \frac{M_{1,2}}{M_*},$$
$$u = \frac{U}{A}, \quad w = \frac{W}{A}, \quad v = \frac{h}{h_*}, \quad p = \frac{PR^2}{2M_*}, \quad s = \frac{H}{2A}. \tag{10}$$

Here M_* and N_* stand for the yield moment and yield force, respectively, for the reference shell of thickness h_*.

Using the notation (10) and assuming that $\lambda \neq 0$ which in turn implies that $\Phi = 0$ one can put the yield condition (6) into the form

$$n_1^2 - n_1 n_2 + n_2^2 + m_1^2 - m_1 m_2 + m_2^2 - v^2 = 0. \tag{11}$$

Note that the case $\lambda = 0$ is associated with the non-deformed shell since according to eqn (8) $\varepsilon_1 = \varepsilon_2 = \kappa_1 = \kappa_2 = 0$ in this case.

Making use of eqn (10), the equilibrium equations (5) and the set [eqns (9)] may be put into the form

$$n_1' = \frac{1}{\rho}(-n_1 + n_2), \quad m_1' = \frac{1}{\rho}(-m_1 + m_2) - \frac{n_1}{s}(z + \rho) - \frac{p}{\rho s}(\rho^2 - \alpha^2),$$
$$w' = z, \quad z' = \frac{z(2m_1 - m_2)}{\rho(2m_2 - m_1)},$$
$$u' = -\tfrac{1}{2}z^2 - \rho z - \frac{zs(2n_1 - n_2)}{\rho(2m_2 - m_1)}, \tag{12}$$

where primes denote the differentiation with respect to ρ and $z = w'$ is an auxiliary variable.

It is easy to recheck that when eliminating λ from eqn (9) one obtains in addition to eqn (12) the relation

$$u(2m_2 - m_1) + sz(2n_2 - n_1) = 0. \tag{13}$$

The inequality constraint (3) will be treated as an equality constraint

$$v - v_0 + \theta^2 = 0, \tag{14}$$

θ being a control variable and $v_0 = h_0/h_*$.

The boundary conditions may be expressed as

$$m_1(\alpha) = n_1(\alpha) = 0, \quad w(\alpha) = w_0 \tag{15}$$

at the inner edge, and as

$$m_1(\beta) = 0, \quad w(\beta) = 0, \quad u(\beta) = 0 \tag{16}$$

at the outer edge, provided the outer edge of the shell is hinged. If, however, the outer edge is simply supported and the radial tension is fixed the condition $u(\beta) = 0$ in eqn (16) must be replaced by the requirement $n_1(\beta) = n$, n being a given constant.

3. OPTIMALITY CONDITIONS

The problem posed above consists of the minimization of the functional (2), taking the differential restrictions (12) and relations (13)–(14) into account. In order to establish the necessary optimality conditions let us introduce the following augmented functional

$$J_* = \int_a^b \left\{ rv + \psi_1 \left[n_1' - \frac{n_2 - n_1}{r} \right] + \psi_2 \left[m_1' - \frac{m_2 - m_1}{r} + \frac{n_1}{s}(r+z) + \frac{p}{rs}(r^2 - a^2) \right] \right.$$
$$+ \psi_3(w' - z) + \psi_4 \left[z' - \frac{z(2m_1 - m_2)}{r(2m_2 - m_1)} \right] + \psi_5 \left[u' + \tfrac{1}{2}z^2 + rz + \frac{sz(2n_1 - n_2)}{r(2m_2 - m_1)} \right]$$
$$+ \varphi(n_1^2 - n_1 n_2 + n_2^2 + m_1^2 - m_1 m_2 + m_2^2 - v^2) + \mu[u(2m_2 - m_1) + sz(2n_2 - n_1)]$$
$$\left. + \eta(v - v_0 + \theta^2) \right\} \mathrm{d}r. \tag{17}$$

Here the variables $\psi_1, \ldots, \psi_5$ stand for the adjoint variables whereas φ, μ, η are certain Lagrangian (non-constant) multipliers. The quantities n_1, m_1, w, z and u are the corresponding state variables whose derivatives stand on the left side of the state equations (12). The rest variables n_2, m_2, v are considered as the control functions.

Note that the state variables are assumed to be continuous at each $\rho \in (\alpha, \beta)$. However, the controls may have finite discontinuities at certain points of the interval (α, β).

Calculating the variation of eqn (17) one obtains the equation

$$\int_a^b \left\{ r\,\delta v + \psi_1\,\delta n_1' - \frac{\psi_1}{r}(\delta n_2 - \delta n_1) + \psi_2\,\delta m_1' - \frac{\psi_2}{r}(\delta m_2 - \delta m_1) + \frac{\psi_2}{s}(r\,\delta n_1 + n_1\,\delta z + z\,\delta n_1) \right.$$
$$+ \psi_3\,\delta w' - \psi_3\,\delta z + \psi_4\,\delta z' - \psi_4\frac{(2m_1 - m_2)\,\delta z}{r(2m_2 - m_1)} - \frac{z\psi_4(3m_2\,\delta m_1 - 3m_1\,\delta m_2)}{r(2m_2 - m_1)^2} + \psi_5\,\delta u'$$
$$+ \psi_5 z\,\delta z + \psi_5 r\,\delta z + \frac{s\psi_5(2n_1 - n_2)\,\delta z}{r(2m_2 - m_1)} + \frac{s\psi_5 z(2\,\delta n_1 - \delta n_2)}{r(2m_2 - m_1)} + \frac{s\psi_5 z(2n_1 - n_2)}{r(2m_2 - m_1)^2}$$
$$\times(-2\,\delta m_2 + \delta m_1) + \varphi[(2n_1 - n_2)\,\delta n_1 + (2n_2 - n_1)\,\delta n_2 + (2m_1 - m_2)\,\delta m_1$$
$$+ (2m_2 - m_1)\,\delta m_2 - 2v\,\delta v] + \mu[(2m_2 - m_1)\,\delta u + 2u\,\delta m_2 - u\,\delta m_1 + s(2n_2 - n_1)\,\delta z$$
$$\left. + sz(2\delta n_2 - \delta n_1)] + \eta(\delta v + 2\theta\,\delta\theta) \right\} \mathrm{d}r = 0. \tag{18}$$

The arbitrariness of the variations of controls v, θ and n_2, m_2, respectively, leads to the equations

$$\eta\theta = 0, \quad \rho - 2\varphi v + \eta = 0 \tag{19}$$

and

$$\begin{gathered} -\frac{y_1}{r} + \frac{s y_5 z}{r(m_1 - 2m_2)} + \varphi(2n_2 - n_1) + 2\mu s z = 0, \\ -\frac{y_2}{r} + \frac{3 z y_4 m_1}{r(2m_2 - m_1)^2} + \frac{2 s y_5 z(n_2 - 2n_1)}{r(2m_2 - m_1)^2} + \varphi(2m_2 - m_1) + 2\mu u = 0. \end{gathered} \tag{20}$$

According to eqn (18) the adjoint set may be written as

$$\begin{aligned} \psi_1' &= \frac{\psi_1}{\rho} + \frac{\psi_2}{s}(z+\rho) + \frac{2 s z \psi_5}{\rho(2m_2 - m_1)} + \varphi(2n_1 - n_2) - \mu s z, \\ \psi_2' &= \frac{\psi_2}{\rho} + \frac{z(s\psi_5(2n_1 - n_2) - 3m_2\psi_4)}{\rho(2m_2 - m_1)^2} + \varphi(2m_1 - m_2) - \mu u, \\ \psi_3' &= 0, \\ \psi_4' &= \frac{n_1}{s}\psi_2 + \frac{\psi_4(m_2 - 2m_1) + s\psi_5(2n_1 - n_2)}{\rho(2m_2 - m_1)} + \psi_5(\rho + z) - \psi_3 + \mu s(2n_2 - n_1), \\ \psi_5' &= \mu(2m_2 - m_1) \end{aligned} \tag{21}$$

and the transversality conditions corresponding to the boundary conditions (15) and (16) become

$$\psi_4(\alpha) = \psi_5(\alpha) = 0, \quad \psi_1(\beta) = \psi_4(\beta) = 0. \tag{22}$$

It is worth noting that in the case where the radial tension is fixed at the outer edge of the shell, i.e. $n_1(\beta) = n$, $u(\beta) \neq 0$, in eqn (22) $\psi_1(\beta) \neq 0$, but $\psi_5(\beta) = 0$.

It follows from eqn (19) that either $\theta = 0$ or $\eta = 0$. In the first case according to eqn (14) and (19) $v = v_0$ and

$$\eta = 2\varphi v_0 - \rho. \tag{23}$$

If, however, $\eta = 0$, $\theta \neq 0$, eqn (19) yields

$$v = \frac{\rho}{2\varphi}. \tag{24}$$

Now the relation (14) shows that $v < v_0$.

Making use of eqn (20) one can define

$$\mu = \frac{\psi_1}{2 s \rho z} - \frac{\psi_5}{2\rho(m_1 - 2m_2)} - \frac{\varphi(2n_2 - n_1)}{2 s z} \tag{25}$$

and

$$\varphi = \frac{-\psi_1 u + \psi_2 sz + \dfrac{3sz^2}{(2m_2-m_1)^2}(sn_1\psi_5 - m_1\psi_4)}{sz\rho\left[2m_2 - m_1 + \dfrac{u}{sz}(n_1 - 2n_2)\right]}. \tag{26}$$

Thus according to eqns (24) and (26) one has

$$v = \frac{\dfrac{\rho^2}{2}(2m_2-m_1)\left(1+\dfrac{u^2}{s^2z^2}\right)}{\psi_2 - \dfrac{u\psi_1}{sz} + \dfrac{3z}{(2m_2-m_1)^2}(sn_1\psi_5 - m_1\psi_4)}, \tag{27}$$

if $v < v_0$.

The equations (11) and (13) combined with eqn (27) permit the determination of the unknown functions n_2, m_2 and v. When defining from eqn (13)

$$n_2 = \frac{1}{2}\left[n_1 - \frac{u}{sz}(2m_2 - m_1)\right] \tag{28}$$

and substituting eqns (27) and (28) into eqn (11) one can obtain the equation

$$\frac{u^2(2m_2-m_1)^2}{4s^2z^2} - \frac{\dfrac{r^4}{4}(2m_2-m_1)^2\left(1+\dfrac{u^2}{s^2z^2}\right)}{\left[y_2 - \dfrac{uy_1}{sz} + \dfrac{3z(sn_1y_5 - m_1y_4)}{(2m_2-m_1)^2}\right]^2} + \tfrac{3}{4}n_1^2 + m_1^2 - m_1m_2 + m_2^2 = 0, \tag{29}$$

which may be used for determination of the quantity m_2.

However, in certain regions the inequality $v < v_0$ may be violated. In these regions eqn (27) must be omitted. Now the relations (11) and (13) could be used for determination of the controls m_2, n_2 bearing in mind that $v = v_0$. Note that eqn (28) holds good but eqn (29) must be replaced by the equation

$$\tfrac{3}{4}n_1^2 + \frac{u^2(2m_2-m_1)^2}{4s^2z^2} + m_1^2 - m_1m_2 + m_2^2 - v_0^2 = 0. \tag{30}$$

4. NUMERICAL RESULTS

The optimization problem set above could be transformed into the boundary value problem with eqns (12) and (21) and boundary conditions (15) and (22), provided $v < v_0$ everywhere. Now, when integrating eqns (12) and (21) one has to take into account the Lagrangian multipliers (25) and (26) as well as the equations for the determination of the control functions (27)–(29) at each mesh point. However, the problem is more complex because in certain regions $v = v_0$.

The whole integration domain is divided into two subdomains D_0 and D_1. It is assumed that for $\rho \in D_0$, $v \equiv v_0$ and for $\rho \in D_1$, $v < v_0$. Note that both of these domains might consist of more than one interval. However, the calculations carried out for the considered cases of the boundary conditions showed that D_0 and D_1 comprise a unique interval (they are both the connected domains). The latter makes the solution procedure simpler. When integrating eqns (12) and (21) in the region D_0, one has to substitute the relations (27) and (29) with $v = v_0$ and eqn (30), respectively. The regions D_0 and D_1 are matched with each other so that the boundary conditions (15) and (22) as well as the continuity requirements

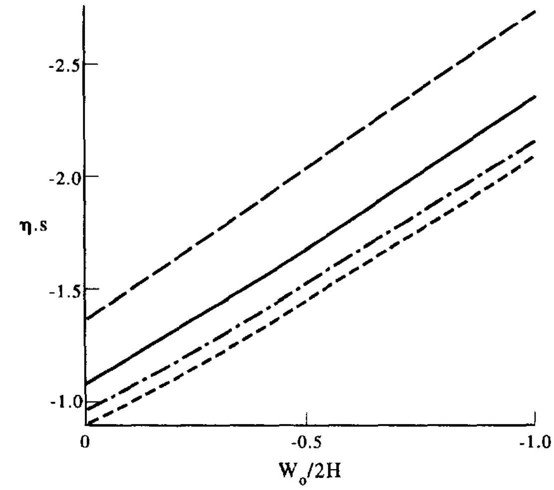

Fig. 2. Load–deflection relations for the shells of constant thickness.

imposed on the state variables n_1, m_1, w, z, u and adjoint variables $\psi_1,\ldots,\psi_5$ at the boundaries of the domains are satisfied.

The corresponding boundary value problems are solved with the modified adjoint operators method. The sets of differential equations are integrated with the aid of the 4th order method of Runge–Kutta.

The calculations are realized in the following manner. In the first stage of the numerical procedure the shell of constant thickness is studied assuming that the maximal deflection w_0 is unspecified. Putting $v = 1$ and solving the boundary value problem with eqns (12) making use of eqns (11) and (13) the stress–strain state of the shell of constant thickness could be determined for each value of the load intensity p. Thus, in particular, one obtains the load–deflection relation $w_0 = w_0(p)$ for the shell of given shape. Naturally, the relation depends on the quantity n, if $u(\beta) \neq 0$ and $n_1(\beta) = n$.

In the final stage of calculations the values of p and w_0 are fixed conformably to the relation $w_0 = w_0(p)$ specified in the previous stage. Here $p \geqslant p_0$, p_0 being the load carrying capacity of the shell. For given values of p, w_0, v_0 the boundary value problems with eqns (12), (15), (21), (22) and (27)–(30), are solved to obtain the optimal thickness distribution for the shell under consideration. At the same time one has to check that the requirement $v(\rho, p) \geqslant v(\rho, p_0)$ is satisfied [Lellep and Majak (1993)].

In order to assess the eventual saving of the material by the utilization of the optimal design let us introduce the following coefficient

$$e = \frac{2}{b^2 - a^2} \int_a^b vr\,\mathrm{d}r, \tag{31}$$

where v stands for the optimal thickness of the shell. Evidently, the economy coefficient (31) is equal to the ratio of the material volumes of the shells with variable and constant thickness h_* respectively.

The results of calculations are presented in Figs 2–10 and in Tables 1–2. Figs 2–10 correspond to the shells with hinged outer edge. In this case the radial displacement vanishes at the edge, i.e. $u(\beta) = 0$.

The load–deflection relations for shells hinged at the outer edge and free at the inner edge are presented in Fig. 2. Here $\alpha = 0.04$; $\beta = 0.1$; $s = 0.05$ and $p_0 \neq 0$, because elastic deformations are neglected. The solid line in Fig. 2 is obtained for the shell of von Mises material (present solution) whereas the dot–dashed line corresponds to the solution obtained by the method of Erkhov (1975). By Erkhov (1975) the original von Mises yield condition is transformed into a piece-wise linear condition which consists of two hexagons

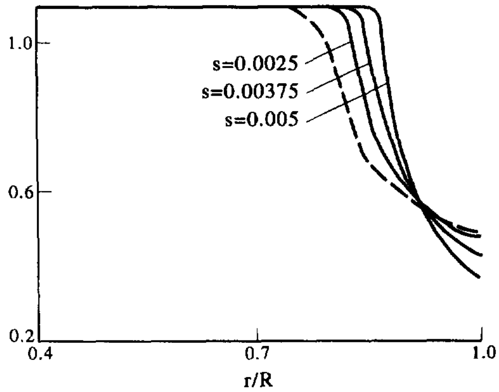

Fig. 3. Optimal thickness of a cap hinged at the outer edge.

lying on the planes of membrane forces and moments, respectively. The dimensions of these hexagons are coupled with each other so that $m^2+n^2 = C$, where $0.75 < C < 1.09$. The upper bound prediction (dashed line in Fig. 2) is obtained making use of the method suggested by Jones (1969) for circular plates. The dotted line in Fig. 2 corresponds to the approach developed by Kondo and Pian (1981). Note that different theoretical predictions do not differ substantially from each other.

The optimal thickness distribution is presented in Fig. 3 for different values of the geometrical parameter s. Here $v_0 = 1.1$. Figs 3–10 correspond to the case of $\alpha = 0.04$ and $\beta = 0.1$. It is worthwhile to emphasize that the optimal thicknesses (solid lines in Fig. 3) are associated with the limit loads, respectively. This implies that the same optimal thickness might be obtained when solving the minimum weight design problem for load carrying capacity. However this is not the case for more shallow shells (if the radius of the shell tends to infinity and s tends to zero). This result is inconsistent with those obtained by the authors for annular and circular plates (Lellep and Majak, 1992), also with the corollaries by Mróz and Gawecki (1975), and Save *et al.* (1989).

The dashed line in Fig. 3 is calculated as the solution of the present problem for $w_0 = 0.3$ and $|p| = 1.428$; $s = 0.0025$. Evidently, the material volume of the latter design is less ($e = 0.895$) than that corresponding to the solid line ($e = 0.924$). However, the solution presented by the dashed line in Fig. 3 must not be called optimal since the equality $v(\rho,p) \geqslant v(\rho,p_0)$ is violated for $\rho \in (0.75;0.93)$. The last inequality is necessary in order to ensure the safety of the design established above [see Lellep and Majak (1993)].

The values of the economy coefficient defined by eqn (31) are accommodated in Table 1. Here the values of the pressure intensity correspond to the load carrying capacity. Thus the material saving is less for deeper shells and greater for more shallow caps (in the range of considered shell parameters).

In order to be convinced of the safety of a design established above one has to carry out the stress–strain analysis for the shell of variable thickness in the post yield point range. The load–deflection relations for the shells with thicknesses presented in Fig. 3 are shown in Fig. 4. Here the solid lines correspond to shells with variable thicknesses whereas dashed lines are associated with the reference shells of constant thickness. The results indicate that the deflections of the shells of variable thickness are less than those obtained for the reference shells of constant thickness for the common value of the pressure intensity.

Table 1. Economy of the design of a spherical cap hinged at the outer edge ($v_0 = 1.1$)

s	0.0025	0.00375	0.005
p	1.074	1.365	1.688
e	0.924	0.936	0.941

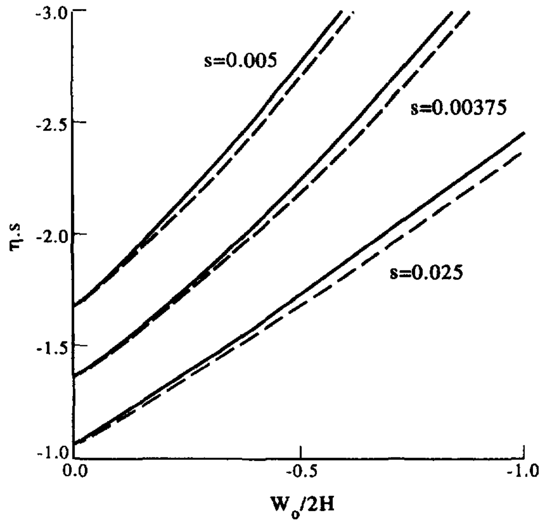

Fig. 4. Load–deflection relations for the shells of variable thickness.

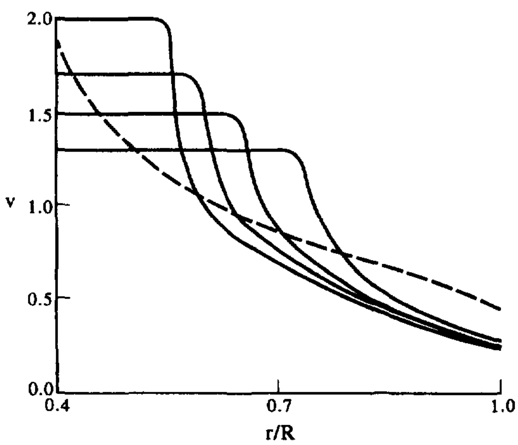

Fig. 5. Optimal thickness distributions corresponding to the different values of v_0.

Evidently, economy of the design established depends on the upper bound imposed on the thickness. The optimal thickness distributions associated with different values of v_0 are presented in Fig. 5 (solid lines). The dashed line in Fig. 5 could be obtained when minimizing the functional

$$I = \int_{\alpha}^{\beta} v^2 \rho \, d\rho. \tag{32}$$

The cost function (32) may be considered as an approximation of the functional

$$I_0 = \max_{\rho} v \tag{33}$$

which is not differentiable. Although eqn (32) has no physical grounds it has been used in the optimal design of plastic shells [see Kruzelecki and Zyczkowski (1985); Lellep and Lepik (1984); Lellep (1991)]. Fig. 5 points out that the cost functions of type (32) with the integrand v^k, where $k \geqslant 2$ could lead to reasonable results even in the cases when the optimal solution of the unconstrained problem with the functional (2) tends to infinity at certain points.

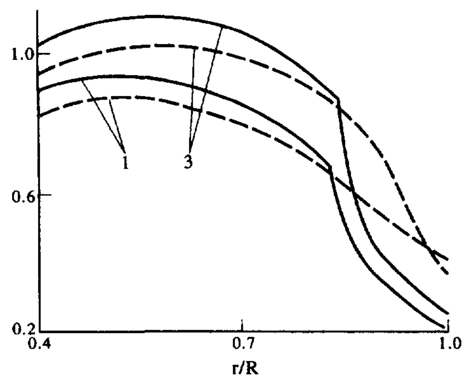

Fig. 6. Hoop force.

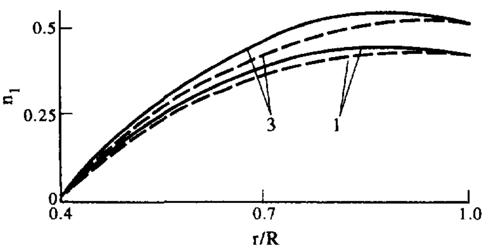

Fig. 7. Radial force.

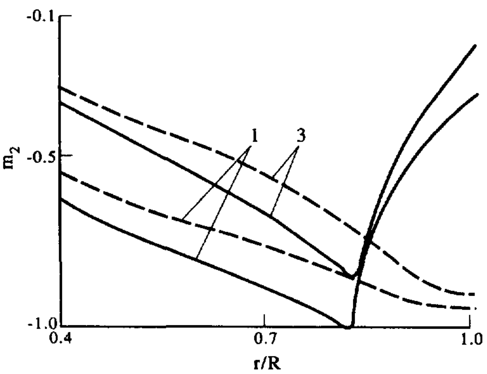

Fig. 8. Hoop moment.

The values of the economy coefficient corresponding to Fig. 5 are presented in Table 2. The last column in Table 2 corresponds to the optimal solution for the problem with the cost function (32), the previous ones to the minimum weight problem with different values of the upper bound on the thickness. Here $s = 0.005$, $p = 1.688$. It is clear from Table 2 that the greater is the upper bound the greater is the amount of the material which can be saved. It is relevant to remark here that the cost function (32) yields satisfactory results in spite of its formal origin.

The stress resultants n_1, n_2, m_1, m_2 are plotted in Figs 6–9 and the radial displacement in Fig. 10. The solid lines in Figs 6–10 correspond to the shell with optimal variable

Table 2. Economy of the design with different upper bounds imposed on the thickness

v_0	1.3	1.5	1.7	2.0	—
e	0.879	0.850	0.832	0.816	0.880

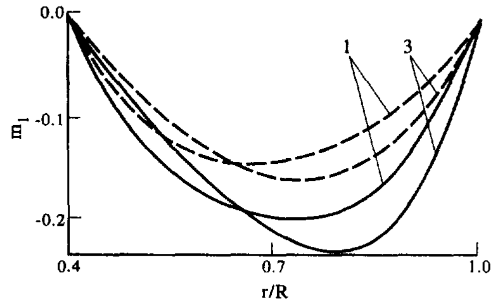

Fig. 9. Radial moment.

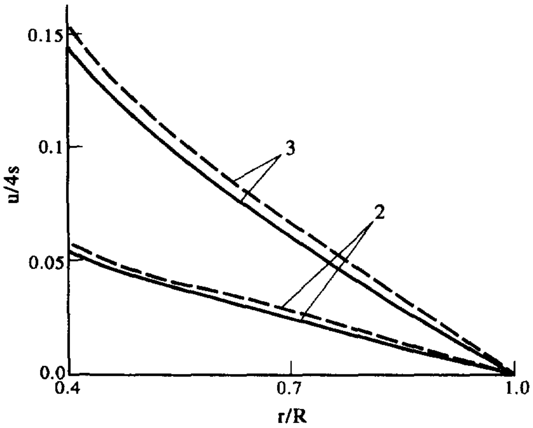

Fig. 10. Meridional displacement.

thickness whereas the dashed lines are associated with the reference shell of constant thickness. Here $s = 0.0025$; $v_0 = 1.1$. The optimal thickness is presented in Fig. 3. The numerical predictions presented in Figs 6–10 are obtained when solving the direct problem for the shell of variable thickness for different levels of the deflection. The curves labelled with 1 in Figs 6–10 are obtained for the limit load ($|p| = 1.074$; $|w_0| = 0$) whereas curves 2 and 3 correspond to the deflections $|w_0|/4s = 0.4$ ($|p| = 1.555$) and $|w_0|/4s = 0.8$ ($|p| = 2.091$), respectively.

Note that the hoop force distributions in the optimal structure differ substantially from those corresponding to the reference shell of constant thickness. However, the discrepancies in the distributions of the radial stresses and displacements are less remarkable. The same phenomenon was observed in the case of axisymmetric plates, as well [see Lellep and Majak (1993)].

5. CONCLUSIONS

An optimal design method has been developed for shallow spherical shells made of an ideally rigid–plastic material. Material of the shells obeys the von Mises yield condition and associated deformation law. The calculations carried out for shells of constant thickness have been compared with the solutions obtained under different assumptions and approximations of the yield surface. Since the theoretical predictions are quite close to each other the use of the present material concept appears to be justified.

Geometrical non-linearity has been taken into account when studying the behaviour of the shells. It appears from a number of comparisons made in this paper that the post-yield stiffness of the shell of variable thickness is greater than that of the reference ("equivalent") shell of constant thickness. This holds good in both cases, if the outer edge is supported so that $u(\beta) = 0$ and so that $u(\beta) \neq 0$, $n_1(\beta) = n$, respectively. In the latter case, loading with relatively small radial tensions must be disregarded. Here the variable thickness may be defined as the optimal thickness corresponding to the minimum weight design for load carrying capacity. Thus the moderately shallow shells where the ratio R/A is not very small are quite favourable structures for the optimization, as the minimum weight designs established for the limit loads are not sensitive to geometrical changes. Note that the behaviour of circular and annular plates of variable thickness is unfavourable in this sense [see Mróz Z and Gawecki A (1975); Lellep and Majak (1993)]. The only exception is the plate subjected to lateral pressure and (not very small) radial tension. Thus the character of the behaviour of the optimized shell depends on the type of the structure, on the material concept and on the loading and support conditions.

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